Math 308L - Autumn 2017
Midterm 2
November 15, 2017

Name: ____________________
Student ID Number: ____________________

Question	Points	Score
1	12	
2	12	
3	12	
4	12	
5	12	
Total:	60	

- There are 5 problems on this exam. Be sure you have all 5 problems on your exam.
- The final answer must be left in exact form. Box your final answer.
- You are allowed the TI-30XIIS calculator. It is possible to complete the exam without a calculator.
- You are allowed a single sheet of 2-sided handwritten self-written notes.
- You must show your work to receive full credit. A correct answer with no supporting work will receive a zero.
- Use the backsides if you need extra space. Make a note of this if you do.
- Do not cheat. This exam should represent your own work. If you are caught cheating, I will report you to the Community Standards and Student Conduct office.

Conventions:

- I will often denote the zero vector by 0.
- When I define a variable, it is defined for that whole question. The A defined in Question 1 is the same for each part.
- I often use x to denote the vector $(x_1, x_2, \ldots, x_n)$. It should be clear from context.
- Sometimes I write vectors as a row and sometimes as a column. The following are the same to me.

$$(1,2,3) \quad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

- I write the evaluation of linear transforms in a few ways. The following are the same to me.

$$T(1,2,3) \quad T((1,2,3)) \quad T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)$$

1. Answer the following parts:

   (a) (6 points) Let

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}.$$

   i. What is A^{-1}?

   ii. What is $\det(2 \cdot A^{-1})$?

   (b) (6 points) (Tricky.) Let

$$B = \begin{bmatrix} 1 & 1 & 11 \\ -1 & 0 & 15 \\ 1 & 2 & 2017 \end{bmatrix}, \quad y = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}.$$

   It turns out that y is in the span of the first and second column of B and B is invertible. What is $B^{-1}y$? (Hint: Despite appearances, this is a quick computation.)

2. Give an example of each of the following. If it is not possible, write "NOT POSSIBLE".

   (a) (3 points) Give an example of 2 linear transforms $T : \mathbb{R}^3 \to \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}^3$ such that $T \circ S : \mathbb{R}^2 \to \mathbb{R}^2$ is invertible.

   (b) (3 points) Give an example of a basis for $\mathbb{R}^3$ such that every basis element lies in the plane $x+y+z = 0$.

   (c) (3 points) Give an example of two different matrices A and B such that $\text{col}(A) = \text{col}(B)$ and $\text{null}(A) = \text{null}(B)$.

   (d) (3 points) Give an example of two 2×2 matrices A and B such that $\det(A+B) \neq \det(A)+\det(B)$.

3. Let $v = (1, 1, -1)$ and $L_v = \text{span}(\{v\})$. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transform that is the projection onto L_v. This tells us 2 things about T:

   - $T(x) = x$ if $x \in L_v$,
   - $T(x) = 0$ if x is orthogonal to v (so if $x \cdot v = 0$).

   There exists a matrix A such that $T(x) = Ax$. The goal of this problem is to determine A.

   (a) (4 points) Give a basis for $\mathbb{R}^3$ that contains v and 2 vectors orthogonal to v. (Hint: Recall that $(a_1, a_2, a_3) \cdot (b_1, b_2, b_3) = a_1b_1 + a_2b_2 + a_3b_3$.)

   (b) (4 points) Answer the following questions about A.

      i. Give a basis for $\text{null}(A)$.

      ii. Give a basis for $\text{col}(A)$.

      iii. What is the rank of A?

      iv. What is $\det(A)$?

   (c) (4 points) What is A? You may express A as a product of matrices and their inverses.

4. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transform defined by $T(x) = Ax$, where A and its reduced echelon form are defined as follows:

$$A = \begin{bmatrix} 1 & 2 & -1 & -3 \\ 2 & 4 & 0 & -4 \\ 3 & 6 & -1 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B.$$

To save time when writing the solutions, let's denote the columns of A by a_1, a_2, a_3, a_4.

(a) (3 points) What is a basis for $\text{row}(A)$?

(b) (3 points) What a basis for the range of T?

(c) (3 points) Write the columns of A corresponding to free variables as a linear combination of pivot columns of A.

(d) (3 points) What is a basis for $\ker(T)$?

5. Let A and B be equivalent matrices given by

$$A = \begin{bmatrix} 2 & 4 & -1 & -2 \\ -1 & -3 & -1 & 0 \\ 1 & 1 & 2 & 2 \\ 2 & 6 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B.$$

Let a_1, a_2, a_3, a_4 be the columns of A. Let $S = \text{span}(\{a_1, a_2\})$ and $T = \text{span}(\{a_3, a_4\})$.

(a) (2 points) What is $\dim(\text{span}(\{a_1, a_2, a_3, a_4\}))$?

(b) (2 points) What is a basis for $\text{null}(A)$?

(c) (2 points) Denote that intersection of S and T by $S \cap T$. This is the subspace of vectors that are in $\text{span}(\{a_1, a_2\})$ **and** in $\text{span}(\{a_3, a_4\})$. What is $\dim(S \cap T)$?

(d) (6 points) (Hard.) What is a basis for $S \cap T$?